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Failure of the Wittmeyer Algorithm for Rational Mean Square Approximation

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Cases are given in which the Wittmeyer algorithm fails to converge to a best rational mean square approximation. In some of these cases, it converges to a nonbest approximation.

The approximations are of the form

$$P(A, x)/Q(B, x) = (a_0 + \cdots + a_n x^n)/(1 + b_1 x + \cdots + b_m x^m) \qquad (1)$$

and the approximation problem is for a continuous f to choose A, B to minimize

$$\int_\alpha^\beta [f(x) - P(A, x)/Q(B, x)]^2 \, dx.$$

This problem has been studied by Cheney and Goldstein [1], who prove the existence of a best approximation.

The Wittmeyer algorithm [2] proceeds as follows. Make a guess B^0 at the denominator of a best approximation and set $k = 0$.

(i) set $w(x) = 1/Q(B^k, x)$,

(ii) find A^{k+1}, B^{k+1} to minimize

$$\delta(A, B) = \int_\alpha^\beta [f(x)\, Q(B, x) - P(A, x)]^2 \, w^2(x) \, dx$$

(iii) add 1 to k and go to (i).

Stage (ii) is a *linear* least square problem and so has a unique solution. This algorithm is similar to an algorithm of Loeb for rational Chebysnev approximation.

LEMMA. *Let f be odd (even), and be approximated by a linear combination of odd and even functions on* $[-\alpha, \alpha]$. *Let there be a unique best approximation to f. Then, it is odd (even).*

Proof. Consider the case when f is odd. Let the best approximation be $O + E$, where O is odd and E is even. By symmetry $O - E$ is also best. By convexity of the set of best approximations, O is best. By uniqueness $E = 0$. The case with f even is similar.

THEOREM. *Let the interval be* $[-\alpha, \alpha]$ *and f be odd, not of the form* (1). *Let n be even and m odd. Let* $Q(B^0, \cdot)$ *be even. Then, the algorithm does not converge to a best approximation.*

Proof. Finding A^1, B^1 involves approximation of an odd function f by

$$f(x) \sum_{k=1}^{m} b_k x^k - P(A, x),$$

with respect to an even weight function. By the lemma, $\sum_{k=1}^{m} b_k{}^1 x^k$ must be even and $P(A^1, \cdot)$ odd. It follows that the next weight is even, hence, all weights are even. Suppose the algorithm converges to A, B then $P(A, \cdot)$ is odd and $Q(B, \cdot)$ is even. Thus, $P(A, \cdot)/Q(B, \cdot)$ is degenerate and by a result of Cheney and Goldstein [1, 239] cannot be best. An inspection of the proof of Cheney and Goldstein's result shows that degenerate approximations are not even locally best.

THEOREM. *Let the interval be* $[-\alpha, \alpha]$ *and f be even, not of the form* (1). *Let n, m be odd. Let* $Q(B^0, \cdot)$ *be even. Then, the algorithm does not converge to a best approximation.*

A similar proof is used.

COROLLARY. *Let* $m = 1$ *and* $Q(B^0, \cdot) = 1$. *Let f be odd (even), not of the form* (1), *and n be even (odd). The algorithm converges to a nonbest approximation.*

Proof. By the argument of the theorems $b_1{}^1 x$ is even, hence, $b_1{}^1 = 0$ and $Q(B^1, \cdot) = 1$. Thus, the algorithm has the same coefficients A, B on all iterations. As the numerator is odd (even), $P(A^1, \cdot)/Q(B^1, \cdot)$ is degenerate and not best.

REFERENCES

1. E. W. CHENEY AND A. A. GOLDSTEIN, Mean square approximation by generalized rational functions, *Math. Z.* **95** (1967), 232–241.
2. L. WITTMEYER, Rational approximation of empirical functions, *BIT* **2** (1962), 53–60.